A CHARACTERIZATION OF THE MOMENTS OF A RANDOM VARIABLE ASSUMING ONLY A FINITE NUMBER OF VALUES

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INTRODUCTION

CONSIDER a random variable X which assumes the values $x_1, x_2, \cdots x_K$ with probabilities $p_1, \cdots p_K$ respectively such that $p_i > 0$ for any $i = 1, 2, \cdots K$ and $\sum_{i=1}^{K} p_i = 1$. Let $\mu_n' = \Sigma\, p_i x_i^n$ be the moment of the n-th order about origin. Then it is plausible to infer that not more than K moments of X should be independent. In the following we show that the moments of X satisfy a linear difference equation with constant coefficients and of the K-th order. These constant co-efficients depend only on $x_1, x_2, \cdots x_K$ and do not involve the probabilities. Using this result we obtain a difference equation for the moments of the random variable X of the above type. It is also shown that the converse is also true. Thus if the moments of a random variable satisfy a linear difference equation with constant coefficients and of order K, then the random variable must be of the above type.

2. Let X be the random variable with probability distribution

$$p\,[X = x_j] = p_j > 0 \quad j = 1, 2, \cdots K \,.\; \sum_{j=1}^{K} p_j = 1. \tag{1}$$

Let $\mu_n' = \sum_{j=1}^{K} p_j x_j^n$ be the n-th order moments about the origin. Consider any set of $(K + 1)$ consecutive moments $(\mu_n', \mu'_{n+1} \cdots \mu'_{n+K})$, we prove that such a set is linearly dependent. More precisely,

Theorem.—We can determine a set of constants $(a_1, a_2, \cdots a_K)$ which depend only on x_i's such that for any integer $n \geqslant 0$ we have

$$\mu'_{n+K} + a_1\mu'_{n+K-1} + \cdots + a_K\mu_n' = 0. \tag{2}$$

Proof.—It means that we have to obtain $a_1, a_2, \cdots a_K$ such that

$$\sum_{j=1}^{K} p_j x_j^{n+K} + a_1 \sum_{j=1}^{K} p_j x_j^{n+K-1} + \cdots + a_K \Sigma p_j x_j^n = 0$$

or

$$\sum_{j=1}^{K} p_j \{x_j^{n+K} + a_1 x_j^{n+K-1} + \cdots + a_K x_j^n\} = 0. \tag{3}$$

This shows that if we choose $(a_1, a_2, \cdots a_K)$ as the solutions of K simultaneous equations, (for all $n \geqslant 0$)

$$x_j^{n+K} + a_1 x_j^{n+K-1} + \cdots + a_K x_j^n = 0 \quad j = 1, 2, \cdots K \tag{4}$$

then whatever be $p_1, p_2 \cdots p_K$ the condition (3) is satisfied.

The above equations are equivalent to

$$x_j^K + a_1 x_j^{K-1} + \cdots + a_K = 0 \qquad j = 1, 2, \cdots K \tag{5}$$

and $(a_1, a_2, \cdots a_K)$ are the solutions of (5). The solutions exist as the determinant of the system of equations is well known Vandermonde's determinant,

$$\Delta = \begin{vmatrix} x_1^{K-1} & x_1^{K-2} & \cdots & 1 \\ x_2^{K-1} & x_2^{K-2} & \cdots & 1 \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ x_K^{K-1} & x_K^{K-2} & \cdots & 1 \end{vmatrix} = \prod_{i<j} (x_i - x_j) \qquad \begin{matrix} i = 1, 2, \cdots K \\ j = 1, 2, \cdots K \end{matrix} \tag{6}$$

As $x_i \neq x_j$ we have $\Delta \neq 0$. Thus the solutions $(a_1, a_2, \cdots a_K)$ exist and are independent of n as well as $(p_1, p_2, \cdots p_K)$. Note that if some $x_i = 0$ we have $a_K = 0$.

Thus we have determined a set of constants $(a_1, a_2, \cdots a_K)$, which depend only on x_i's such that (4) is satisfied for any $n \geqslant 0$, and consequently there exists constants $(a_1, a_2, \cdots a_K)$ such that for $n \geqslant 0$ we have

$$\mu'_{n+K} + a_1 \mu'_{n+K-1} + \cdots + a_K \mu_n' = 0.$$

Thus the moments satisfy a linear difference equation of order K and with constant coefficients. If we use the operator E defined as $E(\mu'_n) = \mu'_{n+1}$ we have

$$\{E^K + a_1 E^{K-1} + \cdots + a_K\} \mu'_n = 0 \quad n \geqslant 0. \tag{7}$$

Corollary.—As an immediate consequence of the above result we observe that μ_K' is expressible in terms of $\mu_0', \mu_1', \cdots \mu'_{K-1}$ and μ'_{K+1} is expressible in terms of $\mu_1', \cdots \mu_K'$, thus μ'_{K+1} is expressible in terms of $\mu_0,' \mu_1,' \cdots \mu'_{K-1}$. In general we can show that μ_n' for any $n \geqslant 0$ is expressible in terms of $\mu_0,' \mu_1,' \cdots \mu'_{K-1}$ and these first K moments may be said to form the base for the sequence of moments.

3. A simple method, for obtaining the linear difference equation (2), can be given with the help of theory of equations.

Suppose for the moment that $a_1, a_2, \cdots a_K$ are known then (4) shows that $x_1, x_2, \cdots x_K$ are the roots of the equation $x^K + a_1 x^{K-1} + \cdots + a_K = 0$ (2′) which is nothing but the characteristic equation of the linear difference equation (2). Thus if we choose

$$a_1 = -\Sigma x_i, \quad a_2 = (-1)^2 \sum_{i \neq j} x_i x_j, \cdots a_K = (-1)^K x_1 x_2 \cdots x_K,$$

equation (4) would be certainly satisfied and the linear difference equation (7) expressed in the terms of symbolic operator E can be written as

$$\left\{ \prod_{j=1}^{K} (E - x_j) \right\} \mu_n' = 0 \qquad n \geqslant 0. \tag{8}$$

It is worth while to note that if $p_1 = p_2 = \cdots p_K = 1/K$ then the linear difference equation (2) is nothing but the Newton's theorem on the sum of the powers of the roots of equation (2′).

Let $\mu_n'(a) = \sum_{j=1}^{K} p_j (x_j - a)^n$ be the n-th order moment about any point a. Then these moments will also satisfy a linear difference equation with constant coefficients and of order K. In fact such an equation is given by

$$\left\{ \prod_{j=1}^{K} [E - (x_j - a)] \right\} \mu_n'(a) = 0 \qquad n \geqslant 0.$$

In particular if a is the mean of the distribution, we get the result that central moments also satisfy a linear difference equation similar to (2).

The above analysis in particular (8) gives us an interesting result. Suppose X and Y are two random variables having the same range of values $x_1, x_2, \cdots x_K$ then their moments satisfy the same linear difference equation (8).

This obviously does not imply that their moments are same, as the moments will be also the functions of probabilities of assuming values $x_1, x_2, \cdots x_K$. This difference will be reflected only in the set of initial conditions when we try to solve the equation (8).

To illustrate the point, let X and Y be binomial variates with

$$P[X=1]=p; \quad P[X=0]=q; \quad P[Y=1]=p'$$

and

$$P[Y=0]\,q'.$$

Then the moments of X and Y both, satisfy the linear difference equation,

$$E(E-1)\mu_n'=0 \qquad n\geqslant 0.$$

$$\mu'_{n+2}-\mu'_{n+1}+0\cdot\mu_n'=0 \qquad n\geqslant 1.$$

But the initial conditions for X are $\mu_0'=1$; $\mu_1'=p$ and those for Y are $\mu'_0=1$; $\mu_1'=p'$.

Thus for X we have $\mu_n'=p$ for $n\geqslant 1$ and for Y $\mu_n'=p'$ for $n\geqslant 1$ and the two are not equal unless $p=p'$, *i.e.*, X and Y are identical.

We may generalize this and say that if the two random variables X and Y have the same range of values $x_1, x_2, \ldots x_K$ and if their first K moments are identical then the variables are also one and the same.

4. Now we consider the converse problem. Let X be the random variable whose moments of all order exist. We further assume that the moments of X satisfy the difference equation (2). Then we want to assert that random variable X takes at most K values.

The linear difference equation is

$$\{E^K+a_1E^{K-1}+\ldots+a_K\}\mu_n'=0. \qquad n\geqslant 0 \tag{2'}$$

and the corresponding characteristic equation is

$$x^K+a_1x^{K-1}+\ldots+a_K=0. \tag{2'}$$

Two cases will have to be considered separately (i) when characteristic equation has no repeated root, (ii) when it has one or more repeated roots.

1*st case.*—In this case characteristic equation has K distinct roots $x_1, x_2, \ldots x_K$ and

$$\mu_n' = c_1 x_1^n + c_2 x_2^n + \ldots + c_K x_K^n \tag{9}$$

where $c_1, c_2, \ldots c_K$ will be determined by the initial conditions. In particular as $\mu_0' = 1$ we have

$$c_1 + c_2 + \ldots + c_K = 1. \tag{10}$$

Let $\phi(t)$ denote the characteristic function (*c.f.*).

$$\phi(t) = \sum_{n=0}^{\infty} \mu_n' \frac{(it)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \{c_1 x_1^n + c_2 x_2^n + \ldots + c_K x_K^n\}$$

$$= \sum_{j=1}^{K} c_j \left\{ \sum_{n=0}^{\infty} \frac{(itx_j)^n}{n!} \right\}$$

$$= \sum_{j=1}^{K} c_j e^{itx_j}. \tag{11}$$

Now $\phi(t)$ is a *c.f.* and as such must be bounded, and hence x_j must be all real, otherwise the function will remain unbounded. As $\phi(t)$ is periodic with period 2π, as $t \to \pm\infty$, $\phi(t)$ does not tend to zero and the corresponding distribution function (*d.f.*) is not everywhere continuous.

In order that (11) must be *c.f.*, it must satisfy a set of sufficient conditions. Following Cramer [1] we have

(*a*) $\phi(t)$ must be bounded and continuous.

(*b*) $\phi(0) = 1$.

(*c*) $\psi(x, A) = \int_0^A \int_0^A \phi(t - u)\, e^{ix(t-u)}\, dt du$.

then $\psi(x, A)$ is real and non-negative for real x and all $A > 0$.

We see that the above conditions (*a*) and (*b*) are satisfied for $\sum_{j=1}^{K} c_j e^{itx_j}$. As $\sum_{j=1}^{K} c_j = 1$ the conditions (*a*) and (*b*) are guaranteed. We observe that $\psi(x_j, A) = c_j A^2$ and will be non-negative if and only if $c_j \geqslant 0$; for $j = 1, 2, \ldots K$ and as $\sum_{j=1}^{K} c_j = 1$ we must have

$$0 \leqslant c_j \leqslant 1 \quad \text{for} \quad j = 1, 2, \ldots K. \tag{12}$$

Thus the *c.f.* $\phi(t)$ has the form $\sum_{j=1}^{K} c_j e^{itx_j}$, where $\sum_{j=1}^{K} c_j = 1$ and $c_j \geqslant 0$.

From this one can immediately conclude that $\phi(t)$ is the *c.f.* of a random variable X assuming values $x_0, x_2, \ldots x_K$ with probabilities $c_1, c_2, \ldots c_K$.

2nd case.—Suppose that characteristic equation admits one root of multiplicity r. Then if x_1 is the root, the solution of the linear difference equation gives

$$\mu_n' = (c_1 + c_2 n + \cdots + c_r n^{r-1})\, x_1^n + c_{r+1} x^n_{r+1} + \cdots + c_K x_K^n \tag{13}$$

where $c_1, c_2, \cdots c_K$ are constants to be determined by initial conditions, and in particular such that

$$\mu_0' = c_1 + c_{r+1} + \cdots + c_K = 1$$

Now

$$\phi(t) = \sum_{n=0}^{\infty} \mu_n' \frac{(it)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left\{ (c_1 + c_2 n + \cdots + c_r n^{r-1}) \frac{(itx_1)^n}{n!} + \sum_{j=r+1}^{K} \frac{(itx_j)^n}{n!} \right\}$$

$$= e^{itx_1} P_{r-1}(itx_1) + \sum_{j=r+1}^{K} c_j e^{itx_j}. \tag{14}$$

where $P_{r-1}(Z)$ denotes a polynomial in Z of $(r-1)$-th degree. It should be noted that constant term in $P_{r-1}(itx_1)$ is c_1. Now $\phi(t)$ consists of two parts one of which $\sum_{j=r+1}^{K} c_j e^{itx_j}$ is bounded. But $P_{r-1}(itx_1)$ is a polynomial of $(r-1)$-th degree in (itx_1) and is not bounded for any t real unless $r = 1$ in which case $P_{r-1}(itx_1) \equiv c_1$. But we have assumed that $r > 1$ hence $P_{r-1}(itx_1)$ is not bounded, and consequently $\phi(t)$ is not bounded. This contradicts our assumption that $\phi(t)$ the *c.f.* exists. This shows that repeated root is not admissible.

This completes the proof of the converse theorem, and shows that the random variable can take at most K-values with non-zero probability.

Summary

It has been shown that the characteristic property of the sequence of moments $\{\mu_n'\}$ of a random variable that assumes only a finite number of values is, that $\{\mu_n'\}$ must satisfy a linear difference equation with constant coefficients and of finite order.

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Reference

1. Cramer, H. .. *Mathematical Methods of Statistics*, 1946.